Properties of Logarithms

Who uses this?

Seismologists use properties of logarithms to calculate the energy released by earthquakes. (See Example 6.)

Lesson Objective(s):

- Use properties to simplify logarithmic expressions.
- Translate between logarithms in any base.

The logarithmic function for pH that you saw in the previous lesson, pH = -log[H +], can also be expressed in exponential form, as 10 -pH = [H +] . Because logarithms are exponents, you can derive the properties of logarithms from the properties of exponents.

Remember that to *multiply* powers with the same base, you add exponents.

$$x^a x^b = x^{a+b}$$

Product Property of Logarithms

For any positive numbers *m*, *n*, and *b* ($b \neq 1$),

WORDS	NUMBERS	ALGEBRA
The logarithm of a product is equal to the sum of the logarithms of its factors.		$\log_x ab = \log_x a + \log_x b$

The property above can be used in reverse to write a sum of logarithms (exponents) as a single logarithm, which can often be simplified.

EXAMPLE 1

Adding Logarithms

Express as a single logarithm. Simplify, if possible.

A $\log_4 2 + \log_4 32$

$\log_4(2 \cdot 32) = \log_4(64)$ $64 = 4^{?}$

$\log_4 4^3 = 3$

B $\log_2 4 + \log_2 16$

$\log_2(4 \cdot 16) = \log_2(64)$ $64 = 2^{?}$

$\log_2 2^6 = 6$

C $\log_3 7x + \log_3 3x$

$\log_3(7x \cdot 3x) = \log_3 21x^2$

Remember that to *divide* powers with the same base, you subtract exponents.

$$\frac{x^a}{x^b} = x^{a-b}$$

Because logarithms are exponents, subtracting logarithms with the same base is the same as finding the logarithm of the quotient with that base.

Quotient Property of Logarithms

For any positive numbers m, n, and $b\ (b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.		$\log_x\left(\frac{a}{b}\right) = \log_x a - \log_x b$

The property above can also be used in reverse.

EXAMPLE 2

Subtracting Logarithms

A Express $\log_2 32 - \log_2 4$ as a single logarithm. Simplify, if possible.

$\log_2\left(\frac{32}{4}\right) = \log_2 8$ $\quad 8 = 2^3 \quad$ $\log_2 2^3 = 3$

B $\log_{1.5} 6.75 - \log_{1.5} 2$

$\log_{1.5}\left(\frac{6.75}{2}\right) = \log_{1.5} 3.375$

C $\log 1000 - \log 10$ $\quad \log\left(\frac{1000}{10}\right) = \log 100 = \log_{10} 10^2 = 2$

Because you can multiply logarithms, you can also take powers of logarithms.

Power Property of Logarithms

For any real number p and positive numbers a and b $(b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a power is the product of the exponent and the logarithm of the base.		$\log_x a^p = p\log_x a$

EXAMPLE 3

Simplifying Logarithms with Exponents

Express as a product. Simplify, if possible.

A $\log_3 81^2$

$2\log_3 81$ $\quad 81 = 3^{\circ}$

$2\log_3 3^4$

$2 \cdot 4$

8

B $\log_5 \left(\frac{1}{5}\right)^3$

$3\log_5\left(\frac{1}{5}\right)$ $\quad \frac{1}{5} = 5^{\circ}$

$3\log_5 5^{-1}$

$3 \cdot -1$

-3

$\boxed{C}$ $\log_5 125^{1/3}$

$\frac{1}{3}\log_5 125$

$\frac{1}{3}\log_5 5^3$ $\quad 125 = 5^{\circ}$

$\frac{1}{3} \cdot 3$

1

Exponential and logarithmic operations undo each other since they are inverse operations.

Inverse Properties of Logarithms and Exponents

For any base b such that $b > 0$ and $b \neq 1$,

ALGEBRA	EXAMPLE
$\log_b b^x = x$	
$b^{\log_b x} = x$	

EXAMPLE 4

Recognizing Inverses

Simplify each expression.

$125 = 5^3$

A $\log_8 8^{3x+1}$

$3x+1$

B $\log_5 125$

$\log_5 5^3 = 3$

C $2^{\log_2 27}$

27

D $\log_3 3^{7+x}$

$7+x$

E $\log_2 16^3$

$3\log_2 16$

$3\log_2 2^4$

$3 \cdot 4$

12

F $3^{\log_3 4.52}$

4.52

Most calculators calculate logarithms only in base 10 or base *e* (see Lesson x-x). You can change a logarithm in one base to a logarithm in another base with the following formula.

Change of Base Formula

For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$ and $b \neq 1$,

ALGEBRA	EXAMPLE
$\log_a b = \frac{\log b}{\log a}$	

EXAMPLE 5

Changing the Base of a Logarithm

A **Evaluate $\log_4 8$.**

$2^3=8$

$2^2=4$

$$\frac{\log 8}{\log 4}$$

$$\frac{\log_2 2^3}{\log_2 2^2} = \frac{3}{2}$$

B $\log_{1/2} 16$

$2^4=16$

$2^{-1}=\frac{1}{2}$

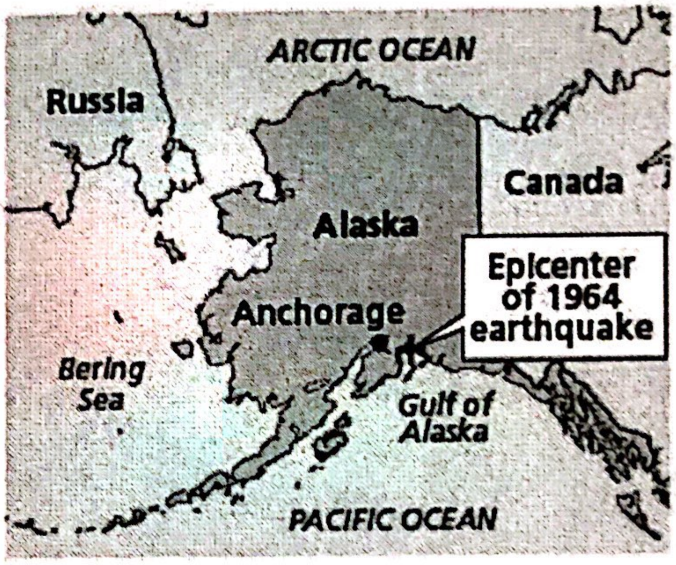
$$\frac{\log 16}{\log \frac{1}{2}}$$

$$\frac{\log_2 2^4}{\log_2 2^{-1}} = \frac{4}{-1} = -4$$

Logarithmic scales are useful for measuring quantities that have a very wide range of values, such as the intensity (loudness) of a sound or the energy released by an earthquake.

XAMPLE 6

Geology Application



Seismologists use the Richter scale to express the energy, or magnitude, of an earthquake. The Richter magnitude of an earthquake, M, is related to the energy released in ergs E shown by the formula $M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$.

In 1964, an earthquake centered at Prince William Sound, Alaska, registered a magnitude of 9.2 on the Richter scale. Find the energy released by the earthquake.

plug in for M

$$\frac{3}{2} \cdot 9.2 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right) \cdot \frac{3}{2}$$

$$13.8 = \log\left(\frac{E}{10^{11.8}}\right)$$

$$13.8 = \log E - \log 10^{11.8}$$

$$13.8 = \log E - 11.8$$

$$+11.8 \qquad\qquad +11.8$$

$$25.6 = \log E$$

$$10^{25.6} = E$$