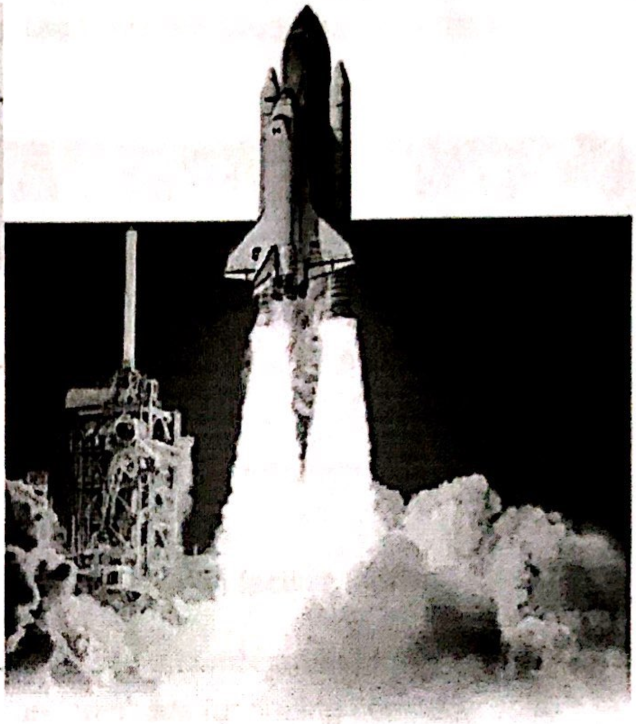
Fundamental Theorem of Algebra

Who uses this?

Aerospace engineers may find roots of polynomial equations to determine dimensions of rockets.(See Example 4.)



Lesson Objective(s):

- Use the Fundamental Theorem of Algebra and its corollary to write a polynomial equation of least degree with given roots.
- Identify all of the roots of a polynomial equation.

You have learned several important properties about real roots of polynomial equations.

The following statements are equivalent:
A real number, r, is a root of the polynomial $P(x)=0$
$P(r)=0$
r is an x-intercept of the graph of $P(x)$
$x-r$ is a factor of $P(x)$
When you divide $P(X)$ by $x-r$, the remainder is 0
r is a zero of $P(x)$

You can use this information to write a polynomial function when given its zeros.

EXAMPLE 1

Writing Polynomial Functions Given Zeros

Write the simplest polynomial function with zeros $-3, \frac{1}{2}$, and 1.

$$P(x) = (x+3)(x-\tfrac{1}{2})(x-1)$$

$$x^2 - \tfrac{1}{2}x + 3x - \tfrac{3}{2}$$

$$(x^2 + \tfrac{5}{2}x - \tfrac{3}{2})(x-1)$$

$$(x^2 + \tfrac{5}{2}x - \tfrac{3}{2})(x-1)$$

$$x^3 + \tfrac{5}{2}x^2 - \tfrac{3}{2}x - x^2 - \tfrac{5}{2}x + \tfrac{3}{2}$$

$$x^3 + \tfrac{3}{2}x^2 - 4x + \tfrac{3}{2}$$

Notice that the degree of the function in Example 1 is the same as the number of zeros. This is true for all polynomial functions. However, all of the zeros are not necessarily real zeros. Polynomial functions, like quadratic functions, may have complex zeros that are not real numbers.

The Fundamental Theorem of Algebra

Every polynomial function of degree $n \geq 1$
number.

Corollary: **of degree $n \geq 1$ has exactly n zeros, including multiplicities.**

Using this theorem, you can write any polynomial function in factored form.

To find all roots of a polynomial equation, you can use a combination of the Rational Root Theorem, the Irrational Root Theorem, and methods for finding complex roots, such as the quadratic formula.

EXAMPLE 2

Finding All Roots of a Polynomial Equation

Solve $x^4 + x^3 + 2x^2 + 4x - 8 = 0$ by finding all roots.

$P = -8 \quad \pm 1, \pm 2, \pm 4, \pm 8$

$Q = 1 \quad \pm 1$

$\frac{P}{q} = \pm 1, \pm 2, \pm 4, \pm 8$

-2	1	1	2	4	-8
	↓	-2	2	-8	8
	1	-1	4	-4	$\varnothing$

$x^3 - x^2 + 4x - 4 = 0$

1	1	-1	4	-4
	↓	1	0	4
	1	0	4	$\varnothing$

$x^2 + 0x + 4 = 0$

$x^2 + 0x + 4$

$a = 1 \quad b = 0 \quad c = 4$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{0 \pm \sqrt{0^2 - 4(1)(4)}}{2(1)}$$

$$\frac{0 \pm \sqrt{-16}}{2} = \frac{0 \pm 4i}{2} = \pm 2i$$

$x = -2, \ x = 1, \ x = 2i, \ x = -2i$

The real numbers are a subset of the complex numbers, so a real number a can be thought of as the complex number $a + 0i$. But here the term *complex root* will only refer to a root of the form $a + bi$, where $b \neq 0$. Complex roots, like irrational roots, come in conjugate pairs. Recall from Chapter x that the complex conjugate of $a + bi$ is $a - bi$.

Complex Conjugate Root Theorem

if $a+bi$ is a root, then $a-bi$ is a root
if $a+b\sqrt{c}$ is a root, then $a-b\sqrt{c}$ is a root

EXAMPLE 3

Writing a Polynomial Function with Complex Zeros

then $-\sqrt{2}$ is a root

A Write the simplest polynomial function with zeros $1 + i$, $\sqrt{2}$, and -3.

then $1-i$ is a root

$$P(x) = (x+3)(x-\sqrt{2})(x+\sqrt{2})(x-(1+i))(x-(1-i))$$

then $-\sqrt{3}$ is a root

B $\sqrt{3}$, $2+i$, 2

then $2-i$ is a root

$$P(x) = (x-\sqrt{3})(x+\sqrt{3})(x-(2+i))(x-(2-i))(x-2)$$